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Symmetry groups and separation of variables of a class of nonlinear diffusion–convection equations

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Abstract. In this paper, a class of nonlinear diffusion–convection equations, $u_t = (D(u)u_x^n)_x + P(u)u_x$, which has quite a large number of physical applications, is analysed by using symmetry group methods which include the classical method, the potential symmetry method and the generalized conditional symmetry method. A complete classification of the functional forms of the diffusion and convection coefficients is presented when the equation admits Lie's point symmetry groups and potential symmetry groups. The separation of variables for the equation is investigated using the generalized conditional symmetry approach. For some interesting cases, exact solutions using the method of separation of variables are discussed in detail.

1. Introduction

This paper is concerned with a class of nonlinear parabolic equations with diffusion and convection terms

$$u_t = (D(u)(u_x)^n)_x + P(u)u_x \tag{1}$$

where $u(x,t)$ is the unknown function, $D(u)$ and $P(u)$ are some smooth functions and the subscripts denote the partial derivatives with respect to the indicated variables. Equation (1) has a wide range of applications (see Atkinson *et al* [1], Esteban *et al* [2] and King [3]). Some important nonlinear partial differential equations (PDEs) are the special cases of equation (1) or its potential form

$$u_t = D(u_x)(u_{xx})^n + \tilde{P}(u_x) \tag{2}$$

where and hereafter $\tilde{P}(u) = \int_0^u P(s)\,\mathrm{d}s$.

In the case of $n = 1$, equation (1) describes the vertical one-dimensional transport of water in homogeneous, non-deformable porous media. When $P = 0$, the group classification using Lie's classical method was presented in Ames [4], Ovisiannikov [5] and Bluman *et al* [6]. When $P \neq 0$, the group classification of equation (1) was given by Oron *et al* [7] and Yung *et al* [8]. An approach for finding nonclassical symmetries (conditional symmetries) involving the reduction of equation (1) was considered by Clarkson *et al* [9], Arrigo *et al* [10], Serov [11] and Galaktionov *et al* [39]. The nonlocal symmetry groups or potential symmetries of equation (1) were studied by Bluman *et al* [12], Akhatov *et al* [13] and Sophocleous [14]. Separation of variables of equation (1) were studied by Dolye [15] using a slightly different method, i.e., the compatibility of differential form. Fokas *et al* [16], Zhdanov [17], Qu [18, 19] and Cherniha [20] discussed the generalized conditional symmetries of equation (1) or its more general form. Some new exact solutions were then obtained, these solutions cannot generally be obtained by the classical and nonclassical methods. Various ansatz-based methods for finding exact solutions of equation (1) were also used in King [21], Fuchchych *et al* [22] and Galaktionov [40]. The homologous property of equation (1) was discussed in [23, 24]. As one can see, this case has been studied thoroughly by many authors.

In the case $n = -1$, equation (1) can be linearized by the point transformations. Equation (2) contains an important nonlinear equation (Bluman *et al* [25], Fokas *et al* [26])

$$u_t = u_x^{-2} u_{xx} + \alpha u_x^{-1}. \tag{3}$$

Equation (3) can be linearized by the hodograph transformations.

For general n, some elegant local and nonlocal equivalence transformation for equation (1) and other types of nonlinear diffusion equations were derived by King [3, 27]. These transformations have been used to obtain new exact solutions and to display the behaviour of solutions for the initial and boundary value problems [27, 28].

When $D(u) = (1 + u_x^2)^{\frac{1-3n}{2}}$, $n \neq 0$, $\tilde{P}(u) = 0$, equation (2) becomes

$$u_t = (1 + u_x^2)^{\frac{1-3n}{2}} (u_{xx})^n \tag{4}$$

which describes the curve evolution in a nonaffine case, we refer to it as the nonaffine curve-shortening equation. It has some important solutions, those solutions are closely related to the symmetry groups of equation (4). Some group-invariant solutions have been applied to study the evolution [29, 30]. When $n = \frac{1}{3}$, equation (4) is then simplified to

$$u_t = (u_{xx})^{1/3}. \tag{5}$$

Equation (5) describes the curve evolution in the affine case, which admits richer symmetry groups (see Ibragimov [31] and Chou *et al* [32]), and is generally referred to as the affine curve-shortening equation. The nonlocal Bäcklund transformation of equation (5) was obtained in [3]. The group-invariant solutions of equation (5) were studied in detail by Chou *et al* [32].

The purpose of the present paper is devoted to the study of the symmetry groups and separation of the variables of equation (1). In section 2, we present a group classification of equation (1) using Lie's classical method, namely, we determine functional forms for the coefficient functions $D(u)$ and $P(u)$ for which different Lie point symmetry groups are admitted. In section 3, we use the method introduced by Bluman *et al* [12] to study the existence of nontrivial potential symmetries of equation (1), the results of papers [12–14] are then extended. Section 4 contains a Lie point symmetry group classification on the potential equation of (1) i.e., equation (2). In section 5, we discuss the separation of variables of equation (1), namely, under what conditions for $D(u)$ and $P(u)$, equation (1) has the separable solution

$$q(u) = f(t)g(x) \tag{6}$$

for some functions $q(u)$. Section 6 concludes with a discussion of our results.

2. A group classification of equation (1)

The classical method for finding symmetry reductions of PDEs is the Lie symmetry group method. To apply Lie's classical method to a kth-order 1+1-dimensional PDE

$$\Delta(x, t, u_t, u_x, u_{xx}, \ldots) = 0 \tag{7}$$

we consider one-parameter transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x, t, u) + \mathrm{O}(\epsilon^2) \\ t^* &= t + \epsilon\tau(x, t, u) + \mathrm{O}(\epsilon^2) \\ u^* &= u + \epsilon\phi(x, t, u) + \mathrm{O}(\epsilon^2) \end{aligned} \tag{8}$$

Table 1. Group classification of equation (1) with $n \neq -1, 1$

$D(u)$	$P(u)$	Infinitesimal generators
Arbitrary	Arbitrary	$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}$
$u^\alpha, \alpha \neq 0$	$\beta \ln u, \beta \neq 0$	$X_1, X_2, X_3 = \frac{\alpha+n-1}{n} t \frac{\partial}{\partial t} + (\frac{\alpha+n-1}{n} x - \beta t) \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$
$u^\alpha, \alpha \neq 0$	$\beta u^\gamma, \beta \neq 0$	$X_1, X_2, X_4 = \frac{\alpha+n-1-\gamma(1+n)}{n} t \frac{\partial}{\partial t} + \frac{\alpha+n-1-\gamma}{n} x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$
$e^{\alpha u}, \alpha \neq 0$	$\beta u, \beta \neq 0$	$X_1, X_2, X_5 = \frac{\alpha}{n} t \frac{\partial}{\partial t} + (\frac{\alpha}{n} x - \beta t) \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
$e^{\alpha u}, \alpha \neq 0$	$\beta e^{\gamma u}, \beta \neq 0$	$X_1, X_2, X_6 = \frac{\alpha-(n+1)\gamma}{n} t \frac{\partial}{\partial t} + \frac{\alpha-\gamma}{n} x \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
$u^\alpha, \alpha \neq 0$	0	$X_1, X_2, X_7 = t \frac{\partial}{\partial t} + \frac{1}{n+1} x \frac{\partial}{\partial x}$
		$X_8(\alpha) = u \frac{\partial}{\partial u} + (1-\alpha-n) t \frac{\partial}{\partial t}$
1	0	$X_1, X_2, X_7, X_8(0), X_9 = \frac{\partial}{\partial u}$
1	$\beta, \beta \neq 0$	$X_1, X_2, X_9, X_{10} = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{n}{n-1} u \frac{\partial}{\partial u}$
$e^{\alpha u}, \alpha \neq 0$	0	$X_1, X_2, X_7, X_{11} = \frac{\alpha}{n} t \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$

where ϵ is the group parameter. Requiring that equation (7) is invariant under the one-parameter group of transformations (8) yields an overdetermined system of ξ, τ and ϕ, which is carried out by setting

$$X^{(k)}(\Delta)|_{\Delta=0} = 0 \tag{9}$$

where X is the corresponding infinitesimal generator of the transformations (8). $X^{(k)}$ is the kth prolongation of X,

$$X^{(k)} = X + \phi^t \frac{\partial}{\partial u_t} + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \cdots$$

where ϕ^t, ϕ^x and ϕ^{xx}, ..., are given explicitly in terms of ξ, τ and ϕ and their derivatives (see [5, 6, 33–35]). From (9), we obtain the following nine determining equations for ξ, τ and ϕ:

$$\begin{aligned}
&\tau_x = \tau_u = 0\\
&\phi_x = \phi_t = 0\\
&\xi_u = \xi_{xx} = 0\\
&\tau_t D + \phi D' + (n-1) D\phi_u - (n+1) D\xi_x = 0\\
&n\phi_u D' + \tau_t D' + \phi D'' - (n+1)\xi_x D' + nD\phi_{uu} = 0\\
&\xi_t + \tau_t P + \phi P' - P\xi_x = 0.
\end{aligned} \tag{10}$$

We do not consider the cases of $n = -1, 1$. If $n = -1$, performing the transformations

$$x \longrightarrow v \qquad u \longrightarrow y$$

equation (1) is transformed to a linear PDE

$$v_t = (D(y)v_y)_y + P(y). \tag{11}$$

The group classification of the case $n = 1$ is given in [7, 8]. Solving system (10), we arrive at the group classification of equation (1) presented in table 1. It is easy to see from table 1 that the maximal dimension of the classical symmetry group for equation (1) is five. Except for translations in time and space and scale transformations, there are no other types of symmetry groups, such as Galilean groups.

3. Potential symmetries of equation (1)

In the case of $n = 1$, the potential symmetries (or nonlocal symmetries) of equation (1) were studied by several authors, such as Bluman *et al* [12] ($P = 0$), Akhatov *et al* [13] ($P = 0$) and

Sophocleous [14] ($P \neq 0$). According to the method of Bluman *et al* [12, 33], if we introduce a potential variable v for a PDE (7), which depends on the global property of u, and write the PDE as the conserved form of further unknown functions, we then obtain a system $Z(x, u, v)$. Any Lie group of transformations for $Z(x, u, v)$ induces a symmetry for equation (7), when at least one of the generators which corresponds to the variables x, u depends explicitly on the potential variable v, the local symmetry of $Z(x, u, v)$ induces a non-local symmetry of equation (7). These normal symmetries are called potential symmetries.

Similar to the approach of [12] by introducing the potential variable v, equation (1) is written as a system of two first-order PDEs

$$u = v_x \qquad v_t = D(u)(u_x)^n + \tilde{P}(u). \tag{12}$$

We determine the infinitesimal transformations of the form

$$\begin{aligned} x^* &= x + \epsilon\xi(x, t, u, v) + \mathrm{O}(\epsilon^2) \\ t^* &= t + \epsilon\tau(x, t, u, v) + \mathrm{O}(\epsilon^2) \\ u^* &= u + \epsilon\phi(x, t, u, v) + \mathrm{O}(\epsilon^2) \\ v^* &= v + \epsilon\psi(x, t, u, v) + \mathrm{O}(\epsilon^2) \end{aligned} \tag{13}$$

which are admitted by equation (1). These transformations induce potential symmetries of equation (1) and point symmetries of equation (2).

It follows from the infinitesimal criterion for invariance of PDEs that equation (12) admits the Lie point group of transformations (13) if and only if

$$V^{(1)}(u - v_x) = 0 \qquad V^{(1)}(v_t - D(u)(u_x)^n - \tilde{P}(u)) = 0 \tag{14}$$

whenever $u = v_x$, and $v_t = D(u)u_x^n + \tilde{P}(u)$, where $V^{(1)}$ is the first extended generator of

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial v} \tag{15}$$

and is given by

$$V^{(1)} = V + \phi^t\frac{\partial}{\partial u_t} + \phi^x\frac{\partial}{\partial u_x} + \psi^t\frac{\partial}{\partial v_t} + \psi^x\frac{\partial}{\partial v_x}.$$

From equations (14) we get an overdetermined system for ξ, τ, ϕ and ψ:

$$\begin{aligned} &\tau_x = \tau_u = \tau_v = \xi_u = 0 \\ &\phi_x + u\phi_v = 0 \\ &\psi_u - u\xi_u = 0 \\ &(\psi_v - \tau_t)D - \xi_v uD - \phi D' - nD(\phi_u - \xi_x) + nuD\xi_v = 0 \\ &\psi_t + (\psi_v - \tau_t)\tilde{P} - u\xi_t - u\tilde{P}\xi_v - \tau_v\tilde{P}^2 - \phi P(u) = 0 \\ &\psi_x - \phi + (\psi_v - \xi_x)u - u^2\xi_v = 0. \end{aligned} \tag{16}$$

Analysis of system (16) leads to the following theorem.

Theorem 1. *Equation (1) admits nontrival potential symmetries if and only if $D(u)$ takes the form*

$$D(u) = (u^2 + pu + q)^{\frac{1-3n}{2}} \exp\left[r\int_0^u \frac{\mathrm{d}s}{s^2 + ps + q}\right] \tag{17}$$

and $\tilde{P}$ satisfies the following first-order ordinary differential equation:

$$\tilde{P}' + \frac{\lambda - u}{u^2 + pu + q}\tilde{P} + \frac{\mu + \kappa u}{u^2 + pu + q} = 0 \tag{18}$$

where p, q, r, λ, μ and κ are constants.

This theorem generalizes the known results of the case $n = 1$ to a more general form. When $n = 1$, $D(u)$ is of the form in [12, 13] and $\tilde{P}$ is of the form in [14]. It is interesting to note that when $r = 0$, and $\tilde{P} = 0$, equation (1) with (17) becomes the nonaffine ($n \neq \frac{1}{3}$) or affine ($n = \frac{1}{3}$) curve-shortening equation.

We do not present the nonlocal symmetries of equation (1) here, and we are more interested in the Lie point symmetries of equation (2) because (2) admits much richer symmetry groups. In fact, the point symmetries of equation (2) are equivalent to nonlocal symmetries of equation (1) from the point of view of obtaining group-invariant solutions.

4. A group classification of equation (2)

Now we consider the group classification of equation (2), i.e., the potential form of equation (1). As for equation (1), (2) is invariant under the one-parameter Lie group of transformations (8) if and only if ξ, τ and ϕ satisfy the following equations:

$$\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})(u_x)^2 - \xi_{uu}(u_x)^3 = 0 \tag{19a}$$

$$\phi_t + (\xi_u u_x^2 - (\phi_u - \xi_x)u_x - \phi_x)P + (\phi_u - \tau_t - \xi_u u_x)\tilde{P} - \xi_t u_x = 0 \tag{19b}$$

$$D'\xi_u(u_x)^2 - D'(\phi_u - \xi_x)u_x - D'\phi_x + D[(3n-1)\xi_u u_x + \phi_u - \tau_t - n\phi_u + 2n\xi_x] = 0 \tag{19c}$$

$$\tau_x = \tau_u = 0. \tag{19d}$$

Equation (19a) is a polynomial of u_x. Equating the coefficients to zero yields the following equations of ξ and ϕ:

$$\phi_{xx} = \xi_{uu} = 0 \tag{20a}$$

$$2\phi_{xu} - \xi_{xx} = 0 \tag{20b}$$

$$\phi_{uu} - 2\xi_{xu} = 0. \tag{20c}$$

From (20a), we find ϕ and ξ are linear functions of x and u, respectively, i.e.,

$$\xi = a_1(x,t)u + a_2(x,u) \qquad \phi = b_1(t,u)x + b_2(t,u). \tag{21}$$

Taking into account (20b) and (20c), we get

$$\begin{aligned} \xi &= (d_2x + f_2)u + d_3x^2 + f_1x + f_3 \\ \phi &= (d_3u + d_5)x + d_2u^2 + d_4u + d_1 \end{aligned} \tag{22}$$

where d_i and f_i, $i = 1, 2, \ldots$, are undetermined functions of t. Substituting (22) into (19b) and (19c), we have the following systems relating d_i, f_i, τ, $D(u)$ and $\tilde{P}(u)$:

$$(d_2u_x + d_3)D' - 2d_2D = 0 \tag{23a}$$

$$(d_2u_x^2 + d_3u_x)D' + [(3n-1)d_2u_x + (3n+1)d_3]D = 0 \tag{23b}$$

$$d_2' = d_3' = 0 \tag{23c}$$

$$d_4' - d_2u_xP - d_3P + 2d_2\tilde{P} - f_2'u_x = 0 \tag{23d}$$

$$d_5' + d_3\tilde{P} + (d_3P - f' - d_2\tilde{P})u_x + d_2Pu_x^2 = 0 \tag{23e}$$

$$[f_2u_x^2 + (f_1 - d_4)u_x - d_5]D' + [(3n-1)f_2u_x + (1-n)d_4 - \tau' + 2nf_1]D = 0 \tag{23f}$$

$$[f_2u_x^2 + (f_1 - d_4)u_x - d_5]D' + (d_4 - \tau' - f_2u_x)\tilde{P} - f_3'u_x + d_6' = 0. \tag{23g}$$

By solving the above system, we obtain the following classification theorem.

Theorem 2. *All possible maximal algebras of invariance of equation (2) for any functions $D(u)$ and $\tilde{P}(u)$ are presented in table 2.*

Table 2. Group classification of equation (2) with $n \neq -1, 1$.

n	$D(u_x)$	$\tilde{P}(u_x)$	Infinitesimal generators
A	A	A	$V_1 = \frac{\partial}{\partial t}$, $V_2 = \frac{\partial}{\partial x}$, $V_3 = \frac{\partial}{\partial u}$
A	A	0	$V_1, V_2, V_3,$ $V_4 = (1+n)t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
$n \neq \frac{1}{3}$	1	0	$V_1, V_2, V_3, V_5(n) = x\frac{\partial}{\partial x} + 2nt\frac{\partial}{\partial t},$ $V_6 = x\frac{\partial}{\partial u}, V_7(n) = u\frac{\partial}{\partial u} + (1-n)t\frac{\partial}{\partial t},$
$n \neq \frac{1}{3}$	1	λe^{-su_x}	$V_1, V_2, V_3,$ $V_8 = s(n+1)t\frac{\partial}{\partial t} + sx\frac{\partial}{\partial x} + (su+nx)\frac{\partial}{\partial u}$
$n \neq \frac{1}{3}$	1	$-\frac{\lambda}{2}u_x^2$	$V_1, V_2, V_3,$ $V_9 = x\frac{\partial}{\partial u} + \lambda t\frac{\partial}{\partial x}$
$n \neq \frac{1}{3}$	1	$(u_x+\mu)^{-\frac{1}{n}}[\lambda - s\ln(u_x+\mu)]$	$V_1, V_2, V_3, V_{10} = (n+1)x\frac{\partial}{\partial x}$ $+[(2n+1)u + n\mu x + snt]\frac{\partial}{\partial u}$
$n \neq \frac{1}{3}$	1	$\lambda(u_x+\mu)^s$	V_1, V_2, V_3 $V_{11} = [(n+1)(\mu-s) - 2(n-1)]t\frac{\partial}{\partial t}$ $+(\mu-s)x\frac{\partial}{\partial x} + [2\mu x + (2+\mu-s)u]\frac{\partial}{\partial u}$
$n \neq \frac{1}{3}$	1	$\lambda\ln(u_x+\mu)$	V_1, V_2, V_3 $V_{12} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (2u+\mu x+\lambda t)\frac{\partial}{\partial u}$
$\frac{1}{3}$	1	$\lambda(u_x^2-\mu^2)^{\frac{1}{2}}(\frac{u_x-\mu}{u_x+\mu})^s$	$V_1, V_2, V_3, V_{13} = 8\mu st\frac{\partial}{\partial t}$ $+(u+6s\mu x)\frac{\partial}{\partial x} + \mu(\mu x+6su)\frac{\partial}{\partial u}$
$\frac{1}{3}$	1	0	$V_1, V_2, V_3, V_5(\frac{1}{3}), V_6, V_7(\frac{1}{3}), V_{14} = u\frac{\partial}{\partial x}$
$\frac{1}{3}$	1	$\frac{1}{2}s - \frac{s}{4\mu}(u_x-\mu)\ln\frac{u_x+\mu}{u_x-\mu}$	$V_1, V_2, V_3, V_{15} = 4\mu t\frac{\partial}{\partial t}$ $+(u+3\mu x)\frac{\partial}{\partial x} + (3u+\mu x+st)\frac{\partial}{\partial u}$
$\frac{1}{3}$	1	$-\frac{1}{2}s - \frac{s}{4\mu}(u_x+\mu)\ln\frac{u_x-\mu}{u_x+\mu}$	$V_1, V_2, V_3, V_{16} = -4\mu t\frac{\partial}{\partial t}$ $+(u+3\mu x)\frac{\partial}{\partial x} + \mu(3u+\mu x-st)\frac{\partial}{\partial u}$
$\frac{1}{3}$	1	$\lambda(u_x^2+\mu^2)^{\frac{1}{2}}e^{s\arctan\frac{u_x}{\mu}}$	$V_1, V_2, V_3, V_{17} = 4\mu st\frac{\partial}{\partial t}$ $+(u+3\mu sx)\frac{\partial}{\partial x} + \mu(3su-\mu x)\frac{\partial}{\partial u}$
$\frac{1}{3}$	1	$\lambda u_x\exp(s/u_x)$	V_1, V_2, V_3 $V_{18} = -4st\frac{\partial}{\partial t} + (u-3sx)\frac{\partial}{\partial x} - 3su\frac{\partial}{\partial u}$
A	D_1	0	V_1, V_2, V_3, V_5 $V_{19} = 2\mu st\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \mu^2 x\frac{\partial}{\partial u}$
A	D_1	$\lambda(u_x^2-\mu^2)^{\frac{1}{2}}(\frac{u_x-\mu}{u_x+\mu})^\sigma$	$V_1, V_2, V_3,$ $V_{20} = \frac{2}{n}\mu((n+1)\sigma - ns)t\frac{\partial}{\partial t}$ $+(u+\frac{2\mu(\sigma-s)}{n}x)\frac{\partial}{\partial x} + (\mu^2 x + \frac{2\mu(\sigma-s)}{n}u)\frac{\partial}{\partial u}$
A	D_1	$\lambda - \frac{\lambda}{2\mu}(u_x-\mu)\ln\frac{u_x+\mu}{u_x-\mu}$	$V_1, V_2, V_3,$ $V_{21} = \frac{\mu}{n}(n+1-2s)t\frac{\partial}{\partial t} + (u+\frac{1-2s}{n}\mu x)\frac{\partial}{\partial x}$ $+\mu(\frac{1-2s}{n}u + \mu x + 2\lambda t)\frac{\partial}{\partial u}$
A	D_1	$-\lambda + \frac{\lambda}{2\mu}(u_x+\mu)\ln\frac{u_x+\mu}{u_x-\mu}$	$V_1, V_2, V_3,$ $V_{22} = -\frac{\mu}{n}(n+1+2s)t\frac{\partial}{\partial t} + (u+\frac{1-2s}{n}\mu x)\frac{\partial}{\partial x}$ $+\mu(\frac{1-2s}{n}u + \mu x + 2\lambda t)\frac{\partial}{\partial u}$
A	D_2	0	V_1, V_2, V_3, V_5 $V_{23} = \mu st\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} - \mu^2 x\frac{\partial}{\partial u}$
A	D_2	$(u_x^2+\mu^2)^{\frac{1}{2}}\exp[\sigma\arctan\frac{u_x}{\mu}]$	$V_1, V_2, V_3,$ $V_{24} = -\frac{\mu}{n}[(n+1)\sigma + s]t\frac{\partial}{\partial t}$ $+[u - \frac{\mu}{n}(\sigma+s)x]\frac{\partial}{\partial x}$ $-[\frac{\mu}{n}(\sigma+s)u + \mu^2 x]\frac{\partial}{\partial u}$
A	D_2	$\lambda u_x\exp(\frac{\sigma}{u_x})$	$V_1, V_2, V_3,$ $V_{25} = [(n+1)\sigma - \mu s]t\frac{\partial}{\partial t}$ $+[nu + (\sigma-\mu s)x]\frac{\partial}{\partial x} + (\sigma-\mu s)u\frac{\partial}{\partial u}$

Table 2. (Continued.)

n	$D(u_x)$	$\tilde{P}(u_x)$	Infinitesimal generators
A	$(u_x-\mu)^s$	0	V_1, V_2, V_3, $V_{26}=(2n+s)t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\mu x\frac{\partial}{\partial u}$ $V_{27}=(1-n-s)t\frac{\partial}{\partial t}+(u-\mu x)\frac{\partial}{\partial u}$
A	$(u_x-\mu)^s$	$\lambda(u_x-\mu)\ln(u_x-\mu)$	$V_1, V_2, V_3, V_{28}=(n+s-1)t\frac{\partial}{\partial t}$ $+[(n-1+s)x-n\lambda t]\frac{\partial}{\partial x}$ $+[(s+2n-1)u-\mu nx+n\lambda\mu t]\frac{\partial}{\partial u}$
A	$(u_x-\mu)^s$	$\lambda\ln(u_x-\mu)$	$V_1, V_2, V_3, V_{29}=(2n+s-1)t\frac{\partial}{\partial t}$ $+(n-1+s)x\frac{\partial}{\partial x}+[(2n+s-1)u$ $-\mu nx-\lambda(n+s-1)t]\frac{\partial}{\partial u}$
A	e^{su_x}	0	V_1, V_2, V_3, V_5 $V_{30}=x\frac{\partial}{\partial u}-st\frac{\partial}{\partial t}$
A	e^{su_x}	$\lambda e^{\sigma u_x}$	V_1, V_2, V_3 $V_{31}=[s-(n+1)\sigma]t\frac{\partial}{\partial t}+(s-\sigma)x\frac{\partial}{\partial x}$ $+[nx+(s-\sigma)u]\frac{\partial}{\partial u}$
A	e^{su_x}	λu_x^2	$V_1, V_2, V_3, V_{32}=st\frac{\partial}{\partial t}$ $+(sx+2\lambda nt)\frac{\partial}{\partial x}+(nx+su)\frac{\partial}{\partial u}$
$n=-\frac{1}{3}$	1	0	$V_1, V_2, V_3, V_5(n), V_6, V_7(n)$, $V_{33}=x^2\frac{\partial}{\partial x}+ux\frac{\partial}{\partial u}$

In table 2, λ, s, μ and σ are some constants. 'A' denotes arbitrary. D_1 and D_2 are given by

$$D_1=(u_x^2-\mu^2)^{\frac{1-3n}{2}}\left(\frac{u_x-\mu}{u_x+\mu}\right)^s$$

and

$$D_2=(u_x^2+\mu^2)^{\frac{1-3n}{2}}\exp\left[s\arctan\frac{u_x}{\mu}\right].$$

From table 2, we see that the maximal dimension of the symmetry groups of equation (2) is seven. Two equations have seven-dimensional symmetry groups. They are equation (5) and the equation

$$u_t=(u_{xx})^{-\frac{1}{3}}. \tag{24}$$

In fact, equation (5) can be transformed into equation (24) by the nonlocal transformations of independent and dependent variables [3, 27]. An interesting feature of equation (5) is that it admits the nonlocal Bäcklund transformation [3]. From the geometric point of view, equation (5) is the simplest and most important nonlinear PDE. It admits an affine group as the symmetry group, and describes the curve evolution in the affine case. It also has important applications in image processing (see Alvarez *et al* [36]). Equation (5) admits much richer symmetry groups than even the standard 1 + 1-dimensional heat equation, as can be seen from the following theorem.

Theorem 3. *An optimal system of symmetry groups of equation (5) is generated by*

$$\begin{aligned}\{&V_1, V_2, V_1+V_3, V_6-V_{14}, V_6-V_{14}\pm V_1, V_5+V_7, V_5+V_7+\alpha(V_6-V_{14})(\alpha\neq 0)\\ &V_6+V_{14}, V_{14}, V_6+V_2, V_6+V_{14}+V_1, V_6\pm V_1, V_6+V_2\pm V_1,\\ &V_5+V_7+V_6+V_{14}V_5+V_7+V_6+V_{14}+V_3,\\ &V_5+V_7+\alpha(V_6+V_{14})(\alpha>0,\alpha\neq 1), V_5+V_7+V_6\}.\end{aligned}$$

Using the above optimal system, we can provide a complete classification for the group invariant solutions. These include travelling wave solutions, spiral wave solutions, eternal solutions, self-similar solutions and eternal-similarity solutions. (For a detailed discussion of these solutions refer to [32].)

5. Separation of variables of equation (1)

We now turn to the discussion of the separation of variables of equation (1). The method used here is the generalized conditional symmetry method developed by Fokas *et al* and Zhdanov (see [16–20]). The case where $n = 1$ and $P = 0$ was discussed by Dolye *et al* [15] who used a slightly different method, i.e., the compatibility of differential forms.

Let us give a brief discussion on the generalized conditional symmetry method. Let K(t,u) denotes a function which depends on a differentiable manner of $u, u_x, u_{xx}, \ldots,$ and t. The function $\sigma(t, x, u)$ is a generalized symmetry of the equation

$$u_t = K(t, u) \tag{25}$$

if and only if

$$\frac{\partial \sigma}{\partial t} = [K, \sigma] \tag{26}$$

where $[K, \sigma] = K'\sigma - \sigma' K$, and the primes denote the Frechet derivative. The concept of conditional symmetry was introduced by Bluman and Cole in [37] under the name of nonclassical symmetry. The generalized conditional symmetries are a generalization of the conditional symmetries as generalized symmetries are a generalization of Lie point symmetries.

Definition 1. *The function $\sigma(t, x, u)$ is a generalized conditional symmetry of equation (25), if there exists a function $F(t, x, u, \sigma)$ such that*

$$\frac{\partial \sigma}{\partial t} = [K, \sigma] + F(t, x, u, \sigma) \qquad F(t, x, u, 0) = 0 \tag{27}$$

where $K(t, u)$ and $\sigma(t, x, u)$ are differentiable functions of t, x and $u, u_x, u_{xx}, \ldots,$ where $F(t, x, u, \sigma)$ is a differentiable function of $t, x, u, u_x, u_{xx}, \ldots,$ and $\sigma, \sigma_x, \sigma_{xx}, \ldots$.

From the definition, we immediately have the following theorem.

Theorem 4. *If σ is independent of t explicitly, then σ is a generalized conditional symmetry of equation (25) if and only if*

$$\sigma' K = 0 \tag{28}$$

whenever $u_t = K$ and $\sigma = 0$.

A significant feature of the generalized conditional symmetry method is that one can use the compatibility of $\sigma = 0$ and the governing equation to get the exact solutions of the considered equation. If equation (25) has a separable solution

$$u = f(t)g(x). \tag{29}$$

After introducing the transformation $v = \ln u$ for equation (25), the derived equation of $v(x, t)$ has a solution of the form

$$v(x, t) = \bar{f}(t) + \bar{g}(x). \tag{30}$$

Equivalently, v satisfies the constraint

$$v_{xt} = 0. \tag{31}$$

To proceed with our discussion, in a manner similar to the approach of [15], we consider a more general PDE

$$v_t = B(v)(v_x)^{n-1}v_{xx} + A(v)(v_x)^{n+1} + Q(v)v_x. \tag{32}$$

When $B = nD$, $A = D'$, equation (32) is just equation (1). Moreover, we have the following theorem.

Theorem 5. *There is a smooth function $v = h(u)$ defined on the entire common domain of the definition of $B(v)$ and $A(v)$ and unique up to affine transformations such that equation (32) has the form (1).*

Proof. Equation (32) can be transformed to equation (1) by transformation $v = h(u)$ if and only if

$$nD = B(h')^{n-1} \qquad D' = A(h')^n + B(h')^{n-2}h'' \qquad P = Q(h(u)). \tag{33}$$

From system (33), we find $h(u)$ can be determined implicitly by

$$\int^{h(u)} \frac{1}{B(s)} \exp\left[\int^s \frac{A(\tilde{s})}{B(\tilde{s})}\,\mathrm{d}\tilde{s}\right] \mathrm{d}s = u$$

and $D(u)$ and $P(u)$ are determined by

$$D(u) = \frac{1}{n}B(h(u))(h')^{n-1}$$

and

$$P(u) = Q(h(u)).$$

Equation (32) has the separable solution (29) if and only if it has the additively separable form (30), because equation (32) preserves the same form under the transformation $v \to \ln v$. Equation (32) has the solution (30) if and only if the solution satisfies (31). □

Theorem 6. *Equation (32) admits the generalized conditional symmetry*

$$\sigma = v_{xt} \tag{34}$$

if and only if $A(v)$, $B(v)$ and $Q(v)$ satisfy the following:

(1) Under $Q = 0$

$$\left(\frac{A'}{B}\right)' = 0 \qquad \left[\frac{(n+1)A + B'}{B}\right]' = 0. \tag{35}$$

(2) Under $Q \neq 0$

$$B = \mathrm{e}^{\alpha v} \qquad A = A_0\mathrm{e}^{\alpha v} \qquad Q = Q_0\mathrm{e}^{\alpha v} \tag{36}$$

where α, A_0 and Q_0 are constants.

Proof. A straightforward calculation shows

$$\sigma' K = \left[(n+1)\left(A' - \frac{AB'}{B}\right) + B'' - \frac{B'^2}{B}\right](v_x)^n v_{xx} + \left(A'' - \frac{B'}{B}A'\right)(v_x)^{n+2} + \left(Q' - \frac{B'}{B}Q\right)v_{xx} + \left(Q'' - \frac{B'}{B}Q'\right)(v_x)^2 \tag{37}$$

where $K = B(v)(v_x)^{n-1}v_{xx} + A(v)(v_x)^{n+1} + Q(v)v_x$. Expression (37) vanishes leading to

$$
\begin{aligned}
&(n+1)\left(A' - \frac{AB'}{B}\right) + B'' - \frac{B'^2}{B} = 0 \\
&A'' - \frac{B'}{B}A' = 0 \qquad Q' - \frac{B'}{B}Q = 0 \qquad Q'' - \frac{B'}{B}Q' = 0
\end{aligned}
\tag{38}
$$

which gives (35) and (36). □

Theorem 7. *Suppose that the coefficient functions of equation (32) satisfy*

$$
\frac{A'}{B} = \alpha \qquad \frac{(n+1)A + B'}{B} = \beta \qquad Q = 0 \tag{39}
$$

where α, β are constants. Then equation (32) has the solution of the form (30) if and only if $\bar{f}(t)$ and $\bar{g}(x)$ satisfy the following system of ordinary differential equations:

$$
\begin{aligned}
&\bar{f}' = B(\bar{f} + \bar{g})(\bar{g}')^{n-1}\bar{g}'' + A(\bar{f} + \bar{g})(\bar{g}')^{n+1} \\
&\bar{g}'\bar{g}''' + (n-1)(\bar{g}'')^2 + \alpha(\bar{g}')^4 + \beta(\bar{g}')^2\bar{g}'' = 0.
\end{aligned}
\tag{40}
$$

Theorem 8. *Suppose that the coefficient functions of equation (32) satisfy*

$$
A = A_0 e^{\alpha v} \qquad B = e^{\alpha v} \qquad Q = Q_0 e^{\alpha v} \tag{41}
$$

where α, A_0 and Q_0 are constants. Then equation (32) has the solution (30) if and only if $\bar{f}$ and $\bar{g}$ satisfy the following ordinary differential equations:

$$
\bar{f}' = \lambda e^{\alpha \bar{f}} \qquad (\bar{g}')^{n-2}\bar{g}'' + A_0(\bar{g}')^n + Q_0 = \lambda e^{-\alpha\bar{g}} \tag{42}
$$

where λ is a constant.

Theorem 9. *Equation (32) admits the generalized conditional symmetry (34), if and only if it is scale equivalent to one of the following equations:*

$$v_t = (v_x)^{n-1}v_{xx} + (v_x)^{n+1} + \lambda v_x \tag{43a}$$

$$v_t = e^v[(v_x)^{n-1}v_{xx} + \delta(v_x)^{n+1} + \lambda v_x] \tag{43b}$$

$$v_t = (e^{(n+1)v} + 1)(v_x)^{n-1}v_{xx} + (v_x)^{n+1} \tag{43c}$$

$$v_t = (e^{(n+1)v} - 1)(v_x)^{n-1}v_{xx} - (v_x)^{n+1} \tag{43d}$$

$$v_t = (1 - e^{-(n+1)v})(v_x)^{n-1}v_{xx} - (v_x)^{n+1} \tag{43e}$$

$$v_t = v e^v (v_x)^{n-1}v_{xx} + \frac{(v-1)e^v}{n+1}(v_x)^{n+1} \tag{43f}$$

$$v_t = (e^{\delta v}\cosh v)(v_x)^{n-1}v_{xx} - e^{\delta v}(\sinh v - \delta\cosh v)(v_x)^{n+1} \qquad \sigma \neq \pm 1 \tag{43g}$$

$$v_t = (e^{\delta v})\sinh v(v_x)^{n-1}v_{xx} - e^{\delta v}(\cosh v - \delta\sinh v)(v_x)^{n+1} \qquad \sigma \neq \pm 1 \tag{43h}$$

$$v_t = (e^{\delta v}\cos v)(v_x)^{n-1}v_{xx} + e^{\delta v}(\sin v + \delta\cos v)(v_x)^{n+1} \qquad -\frac{\pi}{2} < v < \frac{\pi}{2} \tag{43i}$$

where δ and λ are constants.

Proof. By theorems 7 and 8, equation (32) admits the generalized conditional symmetry (34) if and only if A, B and Q satisfy (35) and (36).

If $Q \neq 0$, from (36), we find equation (32) is equivalent to (43a) as $\alpha = 0$, and (43b) as $\alpha \neq 0$.

If $Q = 0$, then A, B and Q satisfy

$$
B = \frac{A'}{\alpha} \qquad A'' - \beta A' + (n+1)\alpha A = 0. \tag{44}
$$

We consider the positive solutions of $B(v)$ for the second-order constant-coefficient ordinary differential equations (44) for all possible values α and β. Using the first equation of (44) to obtain B gives equations (43c)–(43i) and (43a) with $\lambda = 0$. □

Theorem 10. *Equation (1) admits a separation of variables if and only if the coefficient functions $D(u)$ and $P(u)$ are scale equivalent to one of the following functions:*

$$D = u^{1-n} \qquad P = \lambda \tag{45a}$$

$$D = \lambda \exp(\mathrm{e}^{\frac{\lambda}{2}u} + 2u + s) \qquad P = \lambda \mathrm{e}^{2u} + 2s \tag{45b}$$

$$D = \mathrm{e}^{2u}(\mathrm{e}^{2u} + s)^{\frac{2n-2}{3-2n}} \qquad P = \lambda\left(\frac{3-2n}{2}\mathrm{e}^{2u} + s\right)^{\frac{1}{3-2n}} \qquad n \neq \frac{3}{2} \tag{45c}$$

$$D = (u^{\frac{n-1}{n}} - u^2)^{-n} \qquad P = 0 \tag{45d}$$

$$D = (u^2 - u^{\frac{n-1}{n}})^{-n} \qquad P = 0 \tag{45e}$$

$$D = (u^{\frac{n-1}{n}} + u^2)^{-n} \qquad P = 0 \tag{45f}$$

$$D = h^{\frac{2n^2}{n+1}} \mathrm{e}^{\frac{2n}{n+1}h(u)} \qquad \int^{h(u)} \frac{\mathrm{d}z}{\mathrm{e}^{\frac{z}{n+1}} z^{\frac{2n+1}{n+1}}} = u + u_0 \qquad P = 0 \tag{45g}$$

$$D = \mathrm{e}^{\frac{2n}{n+1}\sigma h(u)}[\cosh(h(u))]^{\frac{2n^2}{n+1}} \qquad \int^{h(u)} \frac{\mathrm{d}z}{\mathrm{e}^{\frac{\sigma z}{n+1}}(\cosh z)^{\frac{2n+1}{n+1}}} = u + u_0 \qquad P = 0 \tag{45h}$$

$$D = \mathrm{e}^{\frac{2n}{n+1}\sigma h(u)}[\sinh(h(u))]^{\frac{2n^2}{n+1}} \qquad \int^{h(u)} \frac{\mathrm{d}z}{\mathrm{e}^{\frac{\sigma z}{n+1}}(\sinh z)^{\frac{2n+1}{n+1}}} = u + u_0 \qquad P = 0 \tag{45i}$$

$$D = \mathrm{e}^{\frac{2n}{n+1}\sigma h(u)}[\cos(h(u))]^{\frac{2n^2}{n+1}} \qquad \int^{h(u)} \frac{\mathrm{d}z}{\mathrm{e}^{\frac{\sigma z}{n+1}}(\cos z)^{\frac{2n+1}{n+1}}} = u + u_0 \qquad P = 0 \tag{45j}$$

where λ, s and u_0 are constants.

Proof. Equation (32) is transformed into equation (1) by a transformation of the variable $v = h(u)$ if and only if A, B, D and P satisfy

$$nD = B[h'(u)]^{n-1} \tag{46a}$$

$$D' = A(h')^n + B(h')^{n-2}h''. \tag{46b}$$

From (46a) and (46b), we get

$$\frac{D'}{nD} = \frac{A}{B}h' + \frac{h''}{h'}. \tag{47}$$

The integral to (47) implies

$$D = \exp\left[n\int^{h(u)} \frac{A(z)}{B(z)}\,\mathrm{d}z\right](h')^n. \tag{48}$$

Combining (48) and (46a), we get

$$D = n^{-n}B^n(h(u))\exp\left[\frac{1-n}{n}\int^{h(u)}\frac{A(z)}{B(z)}\,\mathrm{d}z\right] \qquad P(u) = Q(h(u)) \tag{49}$$

where $h(u)$ satisfies

$$h' = \frac{B(h)}{n}\exp\left[-n\int^{h(u)}\frac{A(z)}{B(z)}\,\mathrm{d}z\right].$$

For example, considering equation (43c), A, B and Q are given by

$$B = 1 + \mathrm{e}^{(n+1)v} \qquad A = 1 \qquad Q = 0.$$

Solving system (46), we find that D is scale equivalent to

$$D = (u^{\frac{n-1}{n}} - u^2)^{-n}$$

and the change of variable is

$$v = h(u) = -\frac{1}{n+1}\ln(u^{-\frac{n+1}{n}} - 1)$$

which implies the change of variable $v = h(u)$ transforms equation (43c) to (45d). □

In the same way, we can prove that equations (43a), (43b), (43d),(43e), (43f), (43g),(43h), and (43i) are transformed to (45a), (45b) and (45c), (45e), (45f), (45g), (45h), (45i) and (45j) respectively.

We now use the above results to obtain some exact solutions of equation (1). Some solutions cannot be derived by Lie's classical method.

Example 1. The equation

$$u_t = [(u^2 - u^{\frac{n-1}{n}})^{-n}(u_x)^n]_x \tag{50}$$

is transformed into the equation

$$w_t = (w^2 - w^{1-n})(w_x)^{n-1}w_{xx} - w(w_x)^{n+1} \tag{51}$$

by the change of variable

$$u = (1 - w^{-n-1})^{-\frac{n}{n+1}}. \tag{52}$$

It follows from theorem 9 and the transformation $v = \ln w$, that equation (51) has a separable solution (29) if and only if

$$f' = f^{n+2}[g(g')^{n-1}g'' - (g')^{n+1}] - fg^{-n}(g')^{n-1}g''. \tag{53}$$

The two quantities

$$\rho = g(g')^{n-1}g'' - (g')^{n+1} \qquad \nu = g^{-n}(g')^{n-1}g'' \tag{54}$$

are invariants of the third-order ordinary differential equation

$$g''' = \frac{ng'g''}{g} - (n-1)\frac{(g'')^2}{g'}. \tag{55}$$

We consider three cases as follows:

(i) $\rho = 0$, $\nu \neq 0$.
In this case

$$f(t) = \mathrm{e}^{-t} \qquad g(x) = \mathrm{e}^x.$$

Hence we get the travelling wave solution of equation (50)

$$u = \frac{\mathrm{e}^{n(x-t)}}{[\mathrm{e}^{(n+1)(x-t)} - 1]^{\frac{n}{n+1}}}.$$

This solution can be obtained by the classical method.

(ii) $\rho \neq 0$, $\nu = 0$.
This case leads to the similarity solution of equation (50)

$$u = [1 - (n+1)tx^{-n-1}]^{-\frac{n}{n+1}}$$

which can be obtained by the classical method.

(iii) $\rho \neq 0,\ \nu \neq 0$.
In this case $g(x)$ is implicitly given by

$$\int^{g(x)} \frac{\mathrm{d}s}{(s^{n+1}-1)^{\frac{1}{n+1}}} = x \tag{56}$$

or

$$\int^{g(x)} \frac{\mathrm{d}s}{(s^{n+1}+1)^{\frac{1}{n+1}}} = x \tag{57}$$

or

$$\int^{g(x)} \frac{\mathrm{d}s}{(1-s^{n+1})^{\frac{1}{n+1}}} = x \tag{58}$$

respectively, in terms of the sign of ρ and ν. $f(t)$ is implicitly given by

$$\int^{f(t)} \frac{\mathrm{d}s}{s^{n+2}-s} = t.$$

These solutions cannot be obtained by the classical method.

Example 2. The equation

$$u_t = [(u^2 + u^{\frac{n-1}{n}})^{-n}(u_x)^n]_x \tag{59}$$

is transformed into the equation

$$w_t = (-1)^n[(w^{1-n} - w^2)(w_x)^{n-1}w_{xx} + w(w_x)^{n+1}] \tag{60}$$

by the change of variable

$$u = (w^{-n-1} - 1)^{-\frac{n}{n+1}}. \tag{61}$$

Equation (59) is a generalization of the important Mullins [38] equation

$$u_t = \left(\frac{u_x}{u^2+1}\right)_x. \tag{62}$$

Similar to example 1, equation (60) has a separable solution (29) if and only if

$$f' = (-1)^n[f^{n+2}((g')^{n+1} - g(g')^{n-1}g'') + fg^{-n}(g')^{n-1}g'']. \tag{63}$$

The separable solutions of equation (59) are then given by the following:

(i)

$$f(t) = \mathrm{e}^{(-1)^n t} \qquad g(x) = \mathrm{e}^x.$$

So we get the travelling wave solution

$$u = \frac{\mathrm{e}^{n(x+(-1)^n t)}}{[1 - \mathrm{e}^{(n+1)(x+(-1)^n t)}]^{\frac{n}{n+1}}}.$$

This solution can be obtained by the classical method.

(ii) This case leads to the similarity solution of equation (59)

$$u = [-1 + (-1)^{n-1}(n+1)tx^{-n-1}]^{-\frac{n}{n+1}}$$

which can also be obtained by the classical method.

(iii) In this case $g(x)$ is implicitly given by (56)–(58) respectively, and $f(t)$ is implicitly given by

$$\int^{f(t)} \frac{\mathrm{d}s}{s^{n+2}-s} = (-1)^n t.$$

These solutions cannot be obtained by the classical method.

Example 3. Equation (1) with

$$D(u) = h^{\frac{2n^2}{n+1}} \mathrm{e}^{\frac{2n}{n+1}h} \qquad P(u) = 0 \tag{64}$$

where $h(u)$ is determined implicitly by

$$\int^{h(u)} \frac{\mathrm{d}z}{z^{\frac{2n+1}{n+1}} \mathrm{e}^{\frac{z}{n+1}}} = u$$

is transformed into the equation

$$w_t = w^{2-n} \ln w (w_x)^{n-1} w_{xx} + w^{n-1} \left(-\frac{n}{n+1} \ln w - \frac{1}{n+1} \right) (w_x)^{n+1} \tag{65}$$

by the change of variable

$$u = \int^{\ln w} \frac{\mathrm{d}s}{s^{\frac{2n+1}{n+1}} \mathrm{e}^{\frac{s}{n+1}}}.$$

Equation (65) admits the separable solution (29) if and only if

$$\begin{aligned} f' = f^2 \ln f \left[g^{1-n} (g')^{n-1} g'' - \frac{n}{n+1} g^{-n} (g')^{n+1} \right] \\ + f^2 \left[\left(g^{1-n} (g')^{n-1} g'' - \frac{n}{n+1} g^{-n} (g')^{n+1} \right) \ln g - \frac{1}{n+1} g^{-n} (g')^{n+1} \right]. \end{aligned} \tag{66}$$

The quantities

$$\begin{aligned} \rho &= g^{1-n} (g')^{n-1} g'' - \frac{n}{n+1} g^{-n} (g')^{n+1} \\ \nu &= \left[g^{1-n} (g')^{n-1} g'' - \frac{n}{n+1} g^{-n} (g')^{n+1} \right] \ln g - \frac{1}{n+1} g^{-n} (g')^{n+1} \end{aligned} \tag{67}$$

are two invariants of the equation

$$g''' = (2n-1) \frac{g' g''}{g} + (1-n) \frac{(g'')^2}{g'} - \frac{n^2}{n+1} \frac{(g')^3}{g^2}. \tag{68}$$

We distinguish two cases:

(i) $\rho = 0, \nu \neq 0$.
Without loss of generality, we put $\nu = -1$. This case leads to the similarity solution, which can be obtained by the classical method:

$$w = (n+1)^{-n} \frac{x^{n+1}}{t}.$$

(ii) $\rho \neq 0$.
In this case, $f(t)$ and $g(x)$ are given implicitly by

$$\int^{\ln f} \frac{\mathrm{e}^{-s}}{s+\nu} \mathrm{d}s = t \qquad \int^{\ln g} \frac{\mathrm{e}^{\frac{s}{n+1}}}{(s-\nu)^{\frac{1}{n+1}}} \mathrm{d}s = (n+1)^{\frac{1}{n+1}} x.$$

This solution cannot be obtained by the classical method.

Since equation (1) with $P = 0$ is invariant under transformations

$$t^* = \mathrm{e}^{\epsilon} (t + t_0) \qquad x^* = \mathrm{e}^{(n+1)\epsilon} (x + x_0)$$

if $u = u(x, t)$ is a solution of equation (1), then $v = u(\epsilon(x + x_0), \epsilon^{n+1}(t + t_0))$ is also its solution.

6. Discussion

In this paper, we presented the complete group classification of equation (1) and its potential equation by using Lie's classical method. The existence of nonlocal symmetries was also discussed, and we proved that when the functions of diffusion and convection coefficients satisfy some conditions, there are nontrival nonlocal symmetries for equation (1). Using the generalized conditional symmetry method, we derived a complete list of canonical forms of equation (1) which admit separation of variables in the coordinates. Several examples are considered, and we derived some explicit and implicit exact solutions.

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